Renormalized black hole entropy in anti-de Sitter space via the "brick wall" method

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We consider the entropy of a quantum scalar field on a background black hole geometry in asymptotically anti-de Sitter space-time, using the "brick wall" approach. In anti-de Sitter space, the theory has no infra-red divergences, and all ultra-violet divergences can be absorbed into a renormalization of the coupling constants in the one-loop effective gravitational Lagrangian. We then calculate the finite renormalized entropy for the Schwarzschild-anti-de Sitter and extremal Reissner-Nordström-anti-de Sitter black holes, and show that, at least for large black holes, the entropy is entirely accounted for by the one-loop Lagrangian, apart possibly from terms proportional to the logarithm of the event horizon radius. For small black holes, there are indications that non-perturbative quantum gravity effects become important.

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I. INTRODUCTION

The origin and understanding of black entropy has been a fruitful area of research for nearly three decades, since the original proposal by Bekenstein [1] that the entropy of a black hole should be proportional to the area of the event horizon. Since then, there has been a large body of work attempting to understand the microscopic origin of this black hole entropy. In view of this sizable literature on the subject, we shall not be able to give a complete account of all developments, nor more than a small selection of relevant references. In this article we shall focus on a semi-classical approach, in which the black hole geometry is considered to be a fixed classical background on which quantum fields propagate. This was the approach taken by 't Hooft [2], who considered the entropy of a thermal gas of particles outside the event horizon of a Schwarzschild black hole, using the WKB approximation. The calculation involves divergences coming from the number of modes close to the event horizon, which were regulated by using a "brick wall", namely a cut-off just outside the event horizon.

In the original paper [2], 't Hooft calculated the leading order divergences in the entropy, and found that they were proportional to the area of the event horizon multiplied by $\tilde{\epsilon}^{-2}$, where $\tilde{\epsilon}$ is the proper distance of the "brick wall" from the event horizon (see also [3] for early work on this topic, extending 't Hooft's original results to more general black holes and dimensions other than four). It was subsequently suggested [4,5] that this divergence could be absorbed in a renormalization of Newton's constant. The next stage was to consider the next-to-leading order divergences, which turn out to be proportional to $\log \tilde{\epsilon}$ [6,7]. After some discussion of the interpretation of these terms [7,8], it was agreed that they can be absorbed into a renormalization of the coupling constants in the one-loop effective gravitational Lagrangian, which contains terms quadratic in the curvatures [9–11]. This then accounts for all the divergent terms in the entropy, so the remaining quantity is finite. However, this finite quantity has not received much attention and it is the focus of this paper (see [12] for work in this area in two dimensions).

Subsequently this "brick wall" scenario has been studied by many authors, using various approaches, and agreement has been obtained on the renormalization of the quantum field entropy using the one-loop effective Lagrangian. For example, the "brick wall" approach has been regularized using Pauli-Villars regularization [9,13], the renormalization has been confirmed via a conical singularity approach [10] and the result has been extended to fields with non-zero spin [11].

We should, at this stage, mention that there has been some discussion in the literature as to what this "brick wall" model represents physically. One objection to the model might be that the introduction of the "brick wall" just outside the event horizon is somewhat unphysical. Fortunately, the results outlined above mean that the "brick wall" is simply a useful mathematical tool to regularize the theory, and it can be completely removed by renormalization. The status of the "brick wall" model (with the "brick wall" in place) has been put on a firmer footing recently [14], the problem having been considered in a similar vein previously [15]. Mukhoyama and Israel [14] showed that the

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quantum state studied in the "brick wall" model represents a thermally excited state above the Boulware vacuum, which is the ground state in this scenario. In other words, expectation values of operators calculated for this state are the differences between expectation values in the Hartle-Hawking and Boulware states. This accounts for the divergent contribution to the entropy from close to the event horizon. The difference between the two situations is that, in our case, the divergences can be renormalized away, whereas the divergences in the Boulware state expectation values (such as the expectation value of the stress tensor) arise in quantities which have already been renormalized. However, this work leads us to conclude that the "brick wall" model we shall study in this article is a useful and physically reasonable aid to understanding black hole entropy.

In this article we will use 't Hooft's original "brick wall" approach, and we begin in section II by studying the divergent terms coming from close to the event horizon. We consider a general spherically symmetric black hole in four dimensions, and a minimally coupled scalar field. The effects of rotation [16], higher dimensions [3,17] and non-minimal coupling [18] have been considered elsewhere. We confirm that all the divergences can indeed be renormalized away. We use this approach because it is the most technically straightforward, especially since the divergences are under control. We consider black holes in geometries with a negative cosmological constant, for two reasons:

1. Firstly, the introduction of a negative cosmological constant regularizes the contribution to the entropy from infinity. Although in asymptotically flat space this contribution is well understood as the entropy of a quantum field in flat space, it will simplify our calculations and interpretation if this term is absent.

2. Secondly, black holes in asymptotically anti-de Sitter (adS) space can be in stable equilibrium with a thermal heat bath of radiation at the Hawking temperature, provided that the black hole is sufficiently large [19]. Since we are considering a quantum field in a thermal state, it seems most appropriate to consider black holes for which this configuration is stable.

Our focus in this paper is non-extremal black holes. The application of the "brick wall" model to extremal black holes has been the subject of some controversy [6,13,20–24]. In common with other approaches to black hole entropy, there are two routes to studying extremal black holes. Firstly, one can take the "extremalization after quantization" [23] approach. This is entirely consistent within the "brick wall" model, and yields non-zero results for the entropy (which is the same for other techniques along these lines). The divergent terms are properly accounted for, and we discuss the finite terms in this case in section III A 2. The other approach is "extremalization before quantization" [23]. This is more controversial. It is already known [6,13,21,22,24] that, before an inverse temperature is specified (so that we are "off-shell"), the divergences in the "brick wall" model in this case are more severe than for a non-extremal black hole, and cannot be renormalized away. However, we shall show in section II D that working "on-shell", with vanishing temperature, all the divergent terms are identically zero. In addition, we argue in section III B that the finite terms in the entropy are also vanishing in this approach. This is in agreement with other methods of calculating black hole entropy (for example, a Hamiltonian approach [25]), which give zero entropy for "extremalization before quantization" and non-zero entropy for "extremalization after quantization". We would argue that the answers given by the "brick wall" model are in fact valid only "on shell", i.e. when at the end the temperature has been fixed to be the Hawking temperature, since this is the only natural temperature at which to consider the quantum field. That this outlook gives comparable answers to other methods lends further weight to our approach.

Having shown that, by a suitable renormalization of the coupling constants in the effective gravitational Lagrangian, a finite result for the entropy can be obtained, we then in section III proceed to calculate and interpret this finite entropy. This calculation cannot be performed analytically for a general black hole space-time, so instead we consider the simplest specific black hole geometry in asymptotically adS space, namely Schwarzschild-adS (S-adS). In this section we also restrict attention to a massless scalar field in order to make the integrals more tractable. The introduction of quantum field mass is not expected to alter the qualitative nature of our results.

One subtlety in the renormalization process is exactly which logarithmic terms to discard. Since we can only take the logarithm of a dimensionless quantity, it is necessary to renormalize away a term proportional to $\log(\epsilon/\Upsilon)$, where Υ is some length scale, and ϵ is the co-ordinate distance of the "brick wall" from the event horizon (not to be confused with $\tilde{\epsilon}$, the proper distance from the event horizon). The question is, what is the appropriate length scale Υ in this case? We confirm that for a general choice of length scale Υ, in accordance with the results of other authors [24], the finite entropy contains terms proportional to the logarithm of the radius of the event horizon. Corrections to the Bekenstein-Hawking entropy of this form have been found in other approaches to black hole entropy, for example, using quantum geometry [26]. However, we also argue that there is a natural choice of length scale for black holes in adS for which these terms are absent. For other black holes (not in adS), it remains an open question as to the most appropriate choice of length scale. Having isolated this logarithmic term ambiguity, our focus is then the remaining finite entropy contribution.

For the Schwarzschild-adS black hole, a rather complex expression for the remaining quantum field entropy is obtained. It is not readily apparent that this term arises from the one-loop gravitational Lagrangian, so we firstly

consider the limit in which the radius of the event horizon is much larger than the length scale set by the negative cosmological constant. In other words, we are considering a "large" black hole geometry in which the negative cosmological constant has a great effect on the physics of the situation. This is the limit of greatest interest, since in this limit the semi-classical approximation we are using is most valid, and, furthermore, the black hole can be in stable equilibrium with a thermal bath of radiation at the Hawking temperature which is surrounding the event horizon [19]. In this limit we find that the remaining finite entropy is entirely accounted for by terms coming from the one-loop effective Lagrangian. Therefore, in this limit, we have an entirely consistent picture. The entropy of the black hole, including semi-classical effects, arises from the one-loop effective theory.

We also consider the other limit, namely "small" black holes, for which the radius of the event horizon is much smaller than the length scale set by the cosmological constant. In this case the entropy cannot be explained by the effective Lagrangian, and dimensional considerations suggest that it arises from a Lagrangian proportional to $R^{\frac{7}{2}}$, where R is the Ricci scalar of the geometry. We would expect that for small black holes, higher-order or non-perturbative quantum gravity effects would be significant. Therefore it is not surprising that our semi-classical approach breaks down in this limit and we are led to corrections which involve fractional powers of the curvature.

Next we consider the corresponding calculation for a Reissner-Nordström-adS (RN-adS) black hole, only in the limit in which the black hole is extremal. We follow a "extremalization after quantization" approach, namely the calculation is performed for a non-extremal black hole and then the limit in which the black hole becomes extremal is taken right at the end. The expression for the finite entropy is considerably simpler in this case, and is easily explained as arising from the one-loop Lagrangian, apart from the logarithmic terms, as discussed previously.

If we had taken the "extremalization before quantization" approach, for a massless field we show very easily that the entropy (both finite and divergent terms) must vanish. Therefore we close section III by showing that the same is true if we consider a massive quantum scalar field. The integral required is intractable for a general extremal RN-adS black hole, so we focus on one set of values for the black hole parameters, in which the calculation is simplest. A summary of our results and conclusions can be found in section IV.

II. RENORMALIZATION OF QUANTUM SCALAR FIELD ENTROPY

We now consider the entropy of a quantum scalar field on a general, static, spherically symmetric, black hole background. Firstly, we shall review the standard calculation of the entropy using the WKB approximation, following [2].

A. Entropy calculation using the WKB approximation

We use standard Schwarzschild-like co-ordinates, so that the metric of our background geometry takes the form

$$ds^2 = -\Delta(r)e^{2\delta(r)}dt^2 + \Delta^{-1}dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2. \tag{2.1}$$

Here, and throughout this paper, the metric has signature $(-,+,+,+)$ and we use geometric units in which $G = \hbar = c = k_B = 1$, where k_B is Boltzmann's constant, except in section II C, where Newton's constant will be explicitly retained. The metric function $\Delta(r)$ is given by

$$\Delta(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda r^2}{3}. \tag{2.2}$$

We have included a cosmological constant Λ, which we will usually assume is negative, so that the black hole geometry approaches anti-de Sitter space far from the event horizon (this means that the function $m(r)$ approaches a constant as $r \to \infty$ at infinity). Inclusion of a negative cosmological constant means that the contribution to the entropy far from the black hole is finite. This will enable us to concentrate on the divergences in the entropy arising from the behaviour of the quantum field close to the event horizon, which are our main interest in this section. Using a negative rather than a positive cosmological constant ensures further that there is no cosmological horizon to complicate the issue [21]. We anticipate that our results concerning the renormalization of the divergences arising from close to the event horizon would not be significantly altered if the cosmological constant were zero or positive. At this stage we should comment that the metric (2.1) is in fact the most general for the geometry outside the event horizon (where $\Delta = 0$), when the black hole is static and spherically symmetric, since we do not preclude the possibility that $r = 0$ at the event horizon. Hereafter we shall denote the radius of the event horizon as r_h, bearing in mind that r_h may be zero, as is the case for some extremal black holes in string theory [6,20].

The field equation for a minimally coupled quantum scalar field Φ of mass μ on the background (2.1) is

$$\nabla_\nu \nabla^\nu \Phi - \mu^2 \Phi = 0. \tag{2.3}$$

If Φ has the separable form

$$\Phi(t, r, \theta, \phi) = e^{-iEt} Y_{lm}(\theta, \phi) f(r), \tag{2.4}$$

where $Y_{lm}(\theta, \phi)$ is the usual spherical harmonic function, then the equation for $f(r)$ reads

$$e^{-\delta} \left(r^2 e^{\delta} \Delta f' \right)' + \left[\frac{r^2 E^2}{\Delta} e^{-2\delta} - \mu^2 r^2 - l(l+1) \right] f = 0. \tag{2.5}$$

In order to use the WKB approximation, we define an r-dependent radial wave number $K_{El}(r)$ by

$$K_{El}^2(r) = \frac{E^2}{\Delta^2} e^{-2\delta} - \frac{\mu^2}{\Delta} - \frac{l(l+1)}{r^2 \Delta} \tag{2.6}$$

whenever the right-hand-side is positive, and $K_{El} = 0$ otherwise. By defining a new function $\tilde{f}(r)$ by

$$\tilde{f}(r) = \sqrt{h(r)} f(r), \tag{2.7}$$

where $h(r) = e^{\delta} r^2 \Delta$, the equation (2.5) now has the WKB form:

$$\tilde{f}'' + \left[K_{El}^2(r) + \frac{1}{4h^2} h'^2 - \frac{1}{2h} h'' \right] \tilde{f} = 0. \tag{2.8}$$

The WKB approximation is valid when K_{El}^2 is the dominant term in the square brackets, and will therefore be a particularly good approximation for waves with large E or l. We will regularize our calculations by introducing infra-red and ultra-violet cut-offs, so that $\Phi = 0$ when $r = r_h + \epsilon$ or $r = L$, where $\epsilon \ll 1$ and $L \gg r_h$. Therefore we are placing a "brick wall" a small distance above the event horizon of the black holes. We are also putting the whole system (black hole and quantum field) in a large box, although we shall see shortly that we can in fact take the box to be infinitely large in the situation where we have a negative cosmological constant. The number of radial waves n_{El} satisfies the semi-classical quantization condition

$$\pi n_{El} = \int_{r_h+\epsilon}^{L} K_{El}(r)\, dr, \tag{2.9}$$

and then N_E, the total number of modes with energy less than or equal to E is given by

$$\pi N_E = \int (2l+1) \pi n_{El}\, dl, \tag{2.10}$$

where we have summed over the $(2l+1)$ modes having different values of m for the same l and E, and the ranges of all integrals are restricted by the fact that K_{El} vanishes if the right-hand-side of (2.6) is negative.

The free energy of the quantum scalar field at inverse temperature β is given by

$$e^{-\beta F} = \sum_{\text{modes}} e^{-\beta E} = \prod_{n_{El}, l, m} \frac{1}{1 - e^{-\beta E}}, \tag{2.11}$$

from which we deduce that

$$\begin{aligned} F &= \frac{1}{\beta} \sum_{n_{El}, l, m} \log\left(1 - e^{-\beta E}\right) \\ &= \frac{1}{\beta} \int dl\, (2l+1) \int dn_{El}\, \log\left(1 - e^{-\beta E}\right) \\ &= -\frac{1}{\beta} \int dl\, (2l+1) \int d(\beta E) \frac{n_{El}}{e^{\beta E} - 1} \\ &= -\frac{1}{\pi} \int dl\, (2l+1) \int dE \frac{1}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{L} dr\, K_{El}(r). \end{aligned} \tag{2.12}$$

The l integral can be performed explicitly to give

$$F = -\frac{2}{3\pi}\int \frac{dE}{e^{\beta E}-1}\int_{r_h+\epsilon}^{L} dr \frac{r^2}{\Delta^2} e^{-3\delta}\left[E^2 - \mu^2 \Delta e^{2\delta}\right]^{\frac{3}{2}} = -\frac{2}{3\pi}\int \frac{dE}{e^{\beta E}-1} I(E). \tag{2.13}$$

We close this subsection by considering the contribution to (2.13) for $r \gg 1$. The leading order contribution is:

$$F \sim -\frac{2}{3\pi}\int \frac{dE}{e^{\beta E}-1}\int^{L} dr \frac{9e^{-3\delta}}{\Lambda^2 r^2}\left[E^2 + \frac{1}{3}\mu^2 e^{2\delta}\Lambda r^2\right]^{\frac{3}{2}}. \tag{2.14}$$

If the field is massless ($\mu^2 = 0$), it is clear that this expression tends to zero as $L \to \infty$. On the other hand, if the field is massive this is not so obvious. However, in this case, the requirement that the argument of the square root be positive bounds E from below (since $\Lambda < 0$):

$$E^2 > -\frac{1}{3} e^{2\delta}\Lambda r^2 \mu^2. \tag{2.15}$$

Therefore, when $r \gg 1$, this lower bound on E tends to infinity, which means that the integral over E tends to zero. In either case, the presence of a cosmological constant is crucial. If the field is massive and $\Lambda = 0$, then the lower bound on E becomes $E^2 \geq \mu^2$ for $r \gg 1$, so the E integral does not vanish in this case. The contribution for $r \gg 1$ in the asymptotically flat situation is well understood as the free energy of a quantum field in flat space. However, it clarifies the situation for our purposes to include the negative cosmological constant so that we do not have to consider this part of the free energy any further.

B. Contribution from near the event horizon

We now consider the contribution to the free energy (2.13) from close to the event horizon. For the time being, we focus on the case when the black hole is non-extremal, leaving discussion of the extremal case until section II D. Early work in this area can be found in 't Hooft's original article [2], and also [3], where a general non-extremal black hole near-horizon geometry was transformed into Rindler space.

For a non-extremal black hole, we may expand Δ close to the event horizon as

$$\Delta = \Delta_h'(r-r_h) + \frac{1}{2}\Delta_h''(r-r_h)^2 + O(r-r_h)^3, \tag{2.16}$$

where the subscript h denotes a quantity evaluated at the horizon $r = r_h$. Note that Δ_h' cannot be zero since the black hole is non-extremal. The other quantities in (2.13) can be expanded similarly, so that the contribution to the r integral from close to the event horizon is:

$$\begin{aligned} I(E) &= \int_{r_h+\epsilon} dr\, \frac{E^3 r_h^2 e^{-3\delta_h}}{\Delta_h'^2 (r-r_h)^2}\left[1 + (r-r_h)\left\{\frac{2}{r_h} - 3\delta_h' - \frac{\Delta_h''}{\Delta_h'} - \frac{3}{2}\frac{\Delta_h'}{E^2}\mu^2 e^{2\delta_h}\right\}\right] + O(1) \\ &= \frac{E^3 r_h^2 e^{-3\delta_h}}{\Delta_h'^2}\left[\frac{1}{\epsilon} - \left\{\frac{2}{r_h} - 3\delta_h' - \frac{\Delta_h''}{\Delta_h'} - \frac{3}{2}\frac{\Delta_h'}{E^2}\mu^2 e^{2\delta_h}\right\}\log\epsilon\right] + O(1). \end{aligned} \tag{2.17}$$

In order to perform the E integral, we require the following standard formulae [27]

$$\begin{aligned} \int_0^\infty dE \frac{E^3}{e^{\beta E}-1} &= \frac{\pi^4}{15\beta^4} \\ \int_0^\infty dE \frac{E}{e^{\beta E}-1} &= \frac{\pi^2}{6\beta^2}. \end{aligned} \tag{2.18}$$

Then the contribution to the free energy (2.13) becomes:

$$F = -\frac{2}{3\pi}\frac{r_h^2 e^{-3\delta_h}}{\Delta_h'^2}\left[\frac{1}{\epsilon}\frac{\pi^4}{15\beta^4} - \left\{\frac{2}{r_h} - 3\delta_h' - \frac{\Delta_h''}{\Delta_h'}\right\}\frac{\pi^4}{15\beta^4}\log\epsilon + \frac{\pi^2}{4\beta^2}\mu^2\Delta_h' e^{2\delta_h}\log\epsilon\right] + O(1). \tag{2.19}$$

The entropy S is calculated from the free energy F using the relation

$$S = \beta^2 \frac{\partial F}{\partial \beta}. \tag{2.20}$$

Finally, we then substitute in the value of β, namely the inverse Hawking temperature, so that

$$\beta = \frac{4\pi}{\Delta_h'} e^{-\delta_h}. \tag{2.21}$$

This gives the contribution to the entropy from near the event horizon as

$$S = \frac{1}{\epsilon} \frac{r_h^2}{360} \Delta_h' - \left[\frac{r_h^2}{360} \left(\frac{2\Delta_h'}{r_h} - 3\delta_h' \Delta_h' - \Delta_h'' \right) - \frac{r_h^2 \mu^2}{12} \right] \log \epsilon + O(1). \tag{2.22}$$

The calculation is completed by noting that the variable ϵ is dependent on our choice of co-ordinates. A better cut-off to use is the proper distance of the brick wall from the event horizon, $\tilde{\epsilon}$, which is given by

$$\tilde{\epsilon} = \int_{r_h}^{r_h+\epsilon} dr\, \Delta^{-\frac{1}{2}} = 2\epsilon^{\frac{1}{2}} \left(\Delta_h' \right)^{-\frac{1}{2}} + O(\epsilon^{\frac{3}{2}}), \tag{2.23}$$

in terms of which the entropy is now

$$S = \frac{r_h^2}{90} \tilde{\epsilon}^{-2} - \left[\frac{r_h^2}{180} \left(\frac{2\Delta_h'}{r_h} - 3\delta_h' \Delta_h' - \Delta_h'' \right) - \frac{r_h^2 \mu^2}{6} \right] \log \tilde{\epsilon} + \text{terms finite as } \tilde{\epsilon} \to 0. \tag{2.24}$$

The form (2.24) of the divergent contributions to the entropy is in agreement with previous calculations in the literature [7,9,13,20,21].

C. Renormalization of the entropy

Various mechanisms have been suggested in the literature for regularizing the divergent contribution to the entropy (2.24), such as using Pauli-Villars regularization [9,13]. Here we shall follow a very simple approach, and show that the terms in the entropy which diverge as the cut-off ϵ approaches zero can easily be absorbed in a renormalization of the coupling constants in the one-loop gravitational action.

As is well known, to one-loop the effective gravitational Lagrangian includes quadratic curvature interactions (see, for example, [28]):

$$\mathcal{L} = \frac{1}{16\pi G_B} (R - 2\Lambda_B) + \frac{1}{4\pi} \left[a_B R^2 + b_B R_{\rho\sigma} R^{\rho\sigma} + c_B R_{\rho\sigma\tau\lambda} R^{\rho\sigma\tau\lambda} \right] \tag{2.25}$$

where a_B, b_B and c_B are the (bare) coupling constants for the quadratic interactions and we have included a (bare) Newton's constant G_B. There is also a (bare) cosmological constant Λ_B, but this term does not contribute to the entropy.

The classical gravitational entropy arising from the Lagrangian (2.25) for a black hole having a bifurcate Killing horizon is [29]

$$S = -2\pi \int_{\Sigma} E^{\rho\sigma\tau\lambda} n_{\rho\sigma} n_{\tau\lambda} \tag{2.26}$$

where the integral is performed over the bifurcation surface Σ, with binormal $n_{\rho\sigma}$, and $E^{\rho\sigma\tau\lambda}$ is the functional derivative of $\mathcal{L}$ with respect to $R_{\rho\sigma\tau\lambda}$ holding the metric and connection constant. Taking this functional derivative of (2.25) gives the following entropy [9]:

$$S = \int_{\Sigma} d^2x \sqrt{h} \left[\frac{1}{8G_B} g^{\rho\tau} g^{\sigma\lambda} n_{\rho\sigma} n_{\tau\lambda} + 2a_B R + b_B R_{\rho\sigma} g_{\perp}^{\rho\sigma} - c_B R^{\rho\sigma\tau\lambda} n_{\rho\sigma} n_{\tau\lambda} \right], \tag{2.27}$$

where h is the determinant of the metric on the bifurcation surface and $g_{\perp}^{\rho\sigma}$ is the metric in the normal sub-space to the bifurcation surface. It is straightforward to compute the required curvature components for the metric (2.1), giving the answer

$$S = \frac{1}{4G_B}A_h - 8\pi a_B\left[r_h^2\Delta_h'' + 3r_h^2\Delta_h'\delta_h' + 4r_h\Delta_h' - 2\right]$$
$$-4\pi b_B\left[r_h^2\Delta_h'' + 3r_h^2\Delta_h'\delta_h' + 2r_h\Delta_h'\right] + 8\pi c_B, \qquad (2.28)$$

where $A_h = 4\pi r_h^2$ is the area of the event horizon.

We can now compare this classical entropy with the divergent terms in the entropy of the quantum field on this background, given by (2.24). In section III we shall consider the finite terms in the quantum entropy. Comparing (2.24) and (2.28) it can be seen that the divergences can be absorbed in a renormalization of the coupling constants, as follows:

$$\begin{aligned} G_B^{-1} &\to G_B^{-1} + \frac{1}{90\pi}\tilde{\epsilon}^{-2} - \frac{1}{6\pi}\mu^2\log\tilde{\epsilon} \\ a_B &\to a_B + \frac{1}{720\pi}\log\tilde{\epsilon} \\ b_B &\to b_B - \frac{1}{240\pi}\log\tilde{\epsilon} \\ c_B &\to c_B - \frac{1}{360\pi}\log\tilde{\epsilon}. \end{aligned} \qquad (2.29)$$

Our conclusions are in agreement with the work of other authors [9–11] who considered the renormalization of the leading and sub-leading divergences in the entropy, but using different approaches.

It should be noted at this stage that we have been a little cavalier in our treatment of the $\log\tilde{\epsilon}$ terms. Strictly speaking, one can only take the logarithm of a dimensionless quantity, so we should instead consider $\log(\tilde{\epsilon}/\Upsilon)$, where Υ is some length scale. Although this does not affect the divergence properties of the entropy, it will introduce a term proportional to $\log\Upsilon$ into the finite (renormalized) entropy. This leads to a potential ambiguity in the finite entropy, which will depend on our choice of length scale Υ, and will be discussed further in the next section.

We stress that we have now accounted for all the divergences in the quantum field entropy, since the next order in ϵ contribution will be of order ϵ, and so vanish as $\epsilon \to 0$. Therefore only the finite contribution to the entropy remains. This is the focus of the present work, and is discussed in section III.

D. Extremal black holes

The corresponding entropy calculation for extremal black holes is rather more complex than for non-extremal black holes. The contribution to the entropy from infinity is unchanged, so we concentrate on the contribution from close to the event horizon. In common with the literature, there are two approaches one can take. The first is "extremalization after quantization" [23]. Here one proceeds exactly as for a non-extremal black hole (the calculation in the previous section) and only sets the black hole to be extremal right at the end. This corresponds to considering a whole set of non-extremal black holes, which are approaching an extremal black hole in a limiting process. The issue in this case is at what stage in the calculation one should set the black hole to be extremal. If we follow through the calculation in the non-extremal case, including the (non-zero) temperature and proper distance cut-off $\tilde{\epsilon}$, then the answer is (2.24). Now setting the black hole to be extremal simplifies the result to give

$$S = \frac{r_h^2}{90}\tilde{\epsilon}^{-2} + \left[\frac{\Delta_h'' r_h^2}{180} + \frac{r_h^2\mu^2}{6}\right]\log\tilde{\epsilon} + \text{terms finite as } \tilde{\epsilon}\to 0. \qquad (2.30)$$

In essence, the divergent contributions to the entropy are exactly the same as in the non-extremal case, and may similarly be absorbed in a renormalization of the coupling constants (with the earlier proviso concerning the logarithmic terms). The only situation which is different is when $r_h = 0$, which is the case for some extremal stringy black holes (such as those considered in [6,20]). In this case, as observed in that article, the first divergent term disappears and we are left only with the term proportional to $\log\tilde{\epsilon}$ (note that $\Delta_h'' r_h^2$ will not in general vanish in the limit $r_h \to 0$). This means that for these black holes the entropy is not in fact zero, but will have a contribution from the one-loop corrections to the gravitational Lagrangian.

The second approach is "extremalization before quantization" [23]. Now, we assume from the outset that the black hole is extremal. The expansion of the metric function Δ (2.16) now reads

$$\Delta = \frac{1}{2}\Delta_h''(r-r_h)^2 + \frac{1}{6}\Delta_h'''(r-r_h)^3 + \frac{1}{24}\Delta_h^{(iv)}(r-r_h)^4 + \frac{1}{120}\Delta_h^{(v)}(r-r_h)^5 + O(r-r_h)^6. \qquad (2.31)$$

Here, in order to compute all the divergent contributions to the entropy, it is necessary to continue the expansion for rather more terms than was the case for the non-extremal black hole. The contribution from close to the event horizon to the r integral in the expression for the free energy (2.13) is now rather complex. It has the form

$$I(E) = A_3 E^3 \epsilon^{-3} + A_2 E^3 \epsilon^{-2} + (A_1 E^3 + B_1 E)\epsilon^{-1} + (A_0 E^3 + B_0 E) \log \epsilon + O(1), \tag{2.32}$$

where the A's and B's are numbers depending on r_h, the mass μ of the quantum field and the metric functions Δ and δ and their derivatives at the event horizon, but independent of E. The exact form of the A's and B's is not important for our considerations here. Using the standard integrals (2.18) the contribution to the free energy for a quantum field at inverse temperature β is

$$F = -\frac{2\pi^3}{45\beta^4} \left[A_3 \epsilon^{-3} + A_2 \epsilon^{-2} + A_1 \epsilon^{-1} + A_0 \log \epsilon \right] - \frac{\pi}{9\beta^2} \left[B_1 \epsilon^{-1} + B_0 \log \epsilon \right] + O(1). \tag{2.33}$$

This gives the contribution to the entropy to be

$$S = \frac{8\pi^3}{45\beta^3} \left[A_3 \epsilon^{-3} + A_2 \epsilon^{-2} + A_1 \epsilon^{-1} + A_0 \log \epsilon \right] + \frac{2\pi}{9\beta} \left[B_1 \epsilon^{-1} + B_0 \log \epsilon \right] + O(1). \tag{2.34}$$

This result is in agreement with similar calculations in the literature [13,21,22], giving the leading divergence as proportional to ϵ^{-3}. For an extremal black hole, the proper distance from the event horizon of any point outside the event horizon is infinite, so we have to work only with the co-ordinate dependent cut-off ϵ. In [21], a cut-off proportional to $\log \epsilon$ is used, which results in an exponential divergence in the entropy. This cut-off represents the proper distance from the event horizon, but with an infinite additive factor, the same for all points outside the event horizon, removed. Since we shall shortly show that all the divergent terms in the entropy vanish "on shell", we shall keep ϵ in our calculations.

However, at this stage we are still dealing with an "off-shell" entropy, since we have not specified β. For an extremal black hole, the Hawking temperature vanishes, so that $\beta^{-1} = 0$. Substituting this into (2.34), regardless of the value of our cut-off, we see immediately that all the divergent terms disappear. The entropy of the quantum scalar field on the extremal black hole background, calculated via this approach, does not require any regularization.

III. CALCULATION OF FINITE ENTROPY

A. Massless scalar field

We now turn to the calculation and interpretation of the finite contributions to the entropy of the quantum scalar field. For general black hole geometries, the integral for the free energy (2.13) will not be tractable analytically, but will require a numerical computation. Therefore, in this section, we consider only simple black hole space-times, so that the integrals can be performed exactly. We consider the Schwarzschild-anti-de Sitter and extremal Reissner-Nordström-anti-de Sitter black holes, and, for simplicity, a massless scalar field. Considering black holes which are in asymptotically anti-de Sitter space means that, as observed earlier, the contributions to the free energy and entropy far from the black hole remain finite. We have chosen the simplest examples of an extremal and non-extremal black hole geometry in order to simplify the algebra, and hopefully reveal the essence of the situation.

For both these geometries, the metric function δ vanishes identically, so the integral for the free energy takes the simpler form

$$F = -\frac{2}{3\pi} \int \frac{E^3\, dE}{e^{\beta E} - 1} \int_{r_h+\epsilon}^{L} dr \frac{r^2}{\Delta^2} = -\frac{2\pi^3}{45\beta^4} I_r, \tag{3.1}$$

where we have performed the E integral and

$$I_r = \int_{r_h+\epsilon}^{L} dr \frac{r^2}{\Delta^2}, \tag{3.2}$$

depends only on the geometry of the black hole, and not on E or β (this is the simplification afforded by considering only a massless scalar field). The entropy, calculated via (2.20) is then

$$S = \frac{8\pi^3}{45\beta^3} I_r. \tag{3.3}$$

We now give the result of the computation of the integral I_r in each case.

1. Non-extremal black hole

For the Schwarzschild-adS geometry, the metric function Δ is given by

$$\Delta = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}, \tag{3.4}$$

where M is a constant. We shall simplify the integral I_r by using dimensionless variables as follows:

$$r = r_h x, \qquad \epsilon = \hat{\epsilon} r_h, \qquad L = \hat{L} r_h. \tag{3.5}$$

Then I_r can be written as

$$I_r = \frac{9}{\Lambda^2 r_h} \int_{1+\hat{\epsilon}}^{\hat{L}} dx\, x^4 (x-1)^{-2} (x^2 + x + \xi)^{-2} \tag{3.6}$$

where $\xi = -6M/\Lambda r_h^3$ is a dimensionless positive parameter dependent on the geometry. The integral can be performed exactly, and includes terms proportional to $\hat{\epsilon}^{-1}$ and $\log \hat{\epsilon}$, which, as seen in section II C, can be absorbed in an appropriate renormalization of the coupling constants. However, as mentioned in section II C, there is an inherent ambiguity in this process because of the need to introduce an arbitrary length scale Υ into the logarithm term, so the renormalization of the logarithmic terms needs a little more careful attention.

At this stage in the calculation, we are working with dimensionless variables, the length scale we are using being the radius r_h of the event horizon. Our divergent logarithmic term is therefore

$$\log \hat{\epsilon} = \log(\epsilon / r_h) = \log(\epsilon/\Upsilon) - \log(r_h/\Upsilon), \tag{3.7}$$

where we have introduced a general length scale Υ. What remains in our finite entropy is now, of course, dependent on our choice of length scale Υ, and therefore what we discard with the divergent terms. As observed in [24], for a general Υ, if we discard by renormalization the $\log(\epsilon/\Upsilon)$ term, we are left in the finite entropy with a contribution proportional to $\log(r_h/\Upsilon)$, in other words, proportional to the event horizon radius. Such corrections to the black hole entropy have been found in other contexts, for example, in quantum geometry approaches [26]. However, it should be noted that where $\log r_h$ terms have arisen in the semi-classical entropy [24], this has been for black holes in asymptotically flat space. In that situation, there is only one length scale, the event horizon radius, which is dependent on the black hole geometry, and therefore using that as the length scale Υ does not seem entirely natural (although it does remove the $\log r_h$ contribution to the entropy). In our situation, one might argue that it would be more appropriate to consider discarding a term of the form $\log(\tilde{\epsilon}/\Upsilon)$ rather than $\log(\epsilon/\Upsilon)$ since $\tilde{\epsilon}$ is the proper distance of the "brick wall" from the event horizon, whereas ϵ is a co-ordinate dependent quantity (using $\tilde{\epsilon}$ rather than ϵ makes no difference in asymptotically flat space). Ignoring numerical and non-dimensional geometric factors, using (2.23),

$$\tilde{\epsilon}^2 \propto \epsilon l^2 r_h^{-1} = \hat{\epsilon} l^2, \tag{3.8}$$

where $l = \sqrt{-3/\Lambda}$ is the length scale set by the cosmological constant. Therefore, renormalizing away the term $\log(\tilde{\epsilon}/\Upsilon)$ will leave a contribution to the finite entropy proportional to $\log(l/\Upsilon)$ rather than $\log(r_h/\Upsilon)$. Here, in adS, there are two length scales, r_h and l. It would seem reasonable to suggest that for black holes in adS, the most natural length scale is that set by the cosmological constant Λ, especially as Λ is a constant in the gravitational Lagrangian of the theory, and, further, does not get renormalized in the renormalization of the entropy (see section II C). It could therefore be argued that the term $\log(\tilde{\epsilon}/l)$ is in fact the most natural term to renormalize away, for black holes in adS, as we are considering here. This results in the absence of terms proportional to the logarithm of the event horizon radius in the semi-classical entropy. This is the strategy we shall follow in the remainder of the paper, so that we are investigating the additional part of the finite entropy, which does not include logarithmic terms (possibly containing $\log r_h$ terms) arising due to the ambiguity in choosing an appropriate length scale. However, we should stress that for general black holes *not* in adS, it is not clear what is an appropriate choice of length scale Υ, and that, in general, there will be contributions to the finite entropy proportional to the logarithm of the event horizon radius.

Therefore, we ignore those contributions to the integral (3.6) which are proportional to $\hat{\epsilon}^{-1}$ or $\log \hat{\epsilon}$ and let $\hat{L} \to \infty$ and $\hat{\epsilon} \to 0$ to give our final finite answer, which is:

$$I_r = \frac{9}{\Lambda^2 r_h} (\xi + 2)^{-3} (4\xi - 1)^{-\frac{3}{2}} \mathcal{P}, \tag{3.9}$$

where

$$\mathcal{P} = (2\xi+1)(4\xi-1)^{\frac{3}{2}}\log(\xi+2) + \left(4\xi^4+32\xi^3+12\xi^2+8\xi-2\right)\tan^{-1}\left(\frac{\sqrt{4\xi-1}}{3}\right) + \left(5\xi^3+4\xi^2+\xi-1\right)\sqrt{4\xi-1}. \tag{3.10}$$

The final result for the entropy can now be calculated from (3.3) using the value of the inverse temperature in this geometry, which is:

$$\beta = -\frac{12\pi}{\Lambda r_h}(\xi+2)^{-1}, \tag{3.11}$$

giving

$$S = -\frac{\Lambda r_h^2}{1080}(4\xi-1)^{-\frac{3}{2}}\mathcal{P}. \tag{3.12}$$

Thus we have a rather complicated expression for the exact (apart from logarithmic terms), finite entropy of a massless quantum scalar field on the Schwarzschild-adS black hole, given simply in terms of the parameters governing the geometry. The first term in $\mathcal{P}$ (3.10) involves a logarithm multiplied by a geometric factor which, up to a constant factor independent of the geometry, is the same as the coefficient of the $\log\hat{\epsilon}$ term in the divergent contribution to the entropy (2.28). We may therefore conclude that this term comes from the classical entropy due to a higher-curvature correction to the effective action. However, in this case the coupling constant will contain (or be renormalized by) a factor $\log(\xi+2)$, which is dependent on the geometry, whereas in (2.29) the renormalization factors depended on the cut-off only. This is, in fact, not as serious as it may appear, since the geometric dependence is contained within a logarithmic term. This could easily be absorbed into the $\log\hat{\epsilon}$ divergent term which renormalizes the coupling constants. Further, in any renormalization scheme, there are ambiguities present in absorption (or otherwise) of finite terms, in addition to the ambiguities discussed earlier in this section concerning the logarithmic terms. Here we are primarily interested in the interpretation of the entropy of the quantum field. Therefore we shall not consider the term proportional to $\log(\xi+2)$ any further, since its origin is now understood, and it may be regarded as having been absorbed in a (finite) renormalization of the coupling constants in the higher-curvature Lagrangian (2.25).

The remaining terms in $\mathcal{P}$ are not so easily dealt with. The extra entropy we have is:

$$S_{\text{extra}} = -\frac{\Lambda r_h^2}{1080}(4\xi-1)^{-\frac{3}{2}}\left[\left(4\xi^4+32\xi^3+12\xi^2+8\xi-2\right)\tan^{-1}\left(\frac{\sqrt{4\xi-1}}{3}\right) + \left(5\xi^3+4\xi^2+\xi-1\right)\sqrt{4\xi-1}\right]. \tag{3.13}$$

At first sight, S_{extra} seems to be divergent as $\xi \to 1/4$. Performing the calculation separately when $\xi = 1/4$ gives a finite answer however. In fact the expression (3.13) has a regular limit as $\xi \to 1/4$, as can be seen by using the Taylor expansion

$$\tan^{-1}\left(\frac{\sqrt{4\xi-1}}{3}\right) = \frac{\sqrt{4\xi-1}}{3} - \frac{(4\xi-1)^{\frac{3}{2}}}{27} + O(4\xi-1)^{\frac{5}{2}}. \tag{3.14}$$

With this expansion, the term in square brackets becomes $O(4\xi-1)^{\frac{3}{2}}$, giving a finite answer for S_{extra}.

The extra contribution to the entropy, S_{extra} contains terms which are not proportional to the quantities arising from the higher curvature corrections to the Lagrangian (2.25). All these terms arise from inserting the lower limit $x = 1+\hat{\epsilon}$ in the integral I_r, and then letting $\epsilon \to 0$.

In order to interpret these terms, we first consider the corresponding calculation when $\Lambda = 0$, so we are dealing with the Schwarzschild geometry. In this case the free energy is not finite as $L \to \infty$. However, if we ignore those terms which diverge in this limit (as they are simply the free energy of the scalar field in flat space), together with the terms divergent in ϵ as $\epsilon \to 0$ and any logarithmic terms, we are left with

$$S = -\frac{13}{810}, \tag{3.15}$$

which may be absorbed into the coupling constants in the effective Lagrangian (2.25) in any number of ways, for example:

$$c_B \to c_B + \frac{13}{6480\pi}. \tag{3.16}$$

Therefore the presence of the negative cosmological constant is having a considerable effect on the entropy, although the extra terms arise from contributions near the event horizon.

Including now the cosmological constant, there are two limits of interest. Firstly, consider the case when $-\Lambda r_h^2 \gg 1$. This corresponds to large black holes, whose event horizon radius is large compared with the length scale $l = \sqrt{-3/\Lambda}$ set by the cosmological constant. In this situation, since

$$\xi = -\frac{6M}{\Lambda r_h^3} = 1 - \frac{3}{\Lambda r_h^2} \sim 1 \tag{3.17}$$

the entropy contribution (3.13) becomes, in this limit,

$$S_{\text{extra}} = -\frac{\Lambda r_h^2}{360}\left[\frac{\pi}{\sqrt{3}} + 1\right], \tag{3.18}$$

which arises from the higher derivative terms in the Lagrangian (2.25), and may be absorbed by an appropriate (finite) renormalization of the coupling constants, such as:

$$b_B \to b_B - \frac{1}{720\pi}\left[\frac{\pi}{\sqrt{3}} + 1\right], \tag{3.19}$$

where, again, there is more than one such renormalization which will work.

The second limit is when $-\Lambda r_h^2 \ll 1$, which corresponds to small black holes, having an event horizon radius much smaller than the length scale l set by the cosmological constant. In this limit,

$$\xi \sim -\frac{3}{\Lambda r_h^2} = \frac{l^2}{r_h^2} \gg 1, \tag{3.20}$$

and the dominant contribution to the entropy (3.13) is

$$S_{\text{extra}} = \frac{\pi}{720}\left(\frac{l}{r_h}\right)^3. \tag{3.21}$$

The entropy terms which arise from the higher derivative Lagrangian (2.25) are proportional to $(r_h/l)^2$, so we cannot absorb (3.21) into a renormalization of the coupling constants in (2.25) without making the coupling constants dependent on the geometry in a manner rather more serious than the absorption of the logarithmic terms (although the renormalization of the coupling constants would involve only dimensionless parameters). The question then is whether (3.21) could arise from a gravitational Lagrangian of a different kind.

We will argue only on dimensional grounds. Suppose we have a gravitational Lagrangian which contains nth powers of the curvature, for example

$$\mathcal{L}_n = \alpha_n R^n. \tag{3.22}$$

Then the coupling constant α_n will have dimensions of [length]$^{2n-4}$. If we then calculate the classical entropy (2.26) arising from this Lagrangian, we get a result of the form (the notation as in section II C)

$$\begin{aligned} S &= \int_\Sigma d^2x\, \sqrt{h}\, n\alpha_n R^{n-1} \\ &= 4\pi n\alpha_n r_h^{4-2n}(-1)^{n-1}\left[r_h^2\Delta_h'' + 3r_h^2\Delta_h'\delta_h' + 4r_h\Delta_h' - 2\right]^{n-1}. \end{aligned} \tag{3.23}$$

We now specialize to the Schwarzschild-adS geometry, with

$$\xi \sim \frac{l^2}{r_h^2} \gg 1. \tag{3.24}$$

We obtain

$$S \sim \alpha_n r_h^{4-2n}, \tag{3.25}$$

where all the factors of l have canceled. It seems reasonable in this model to have α_n proportional to l^{2n-4}, since l is the length scale set by the coupling constant $-\Lambda$ of the theory. Therefore

$$S \sim \left(\frac{l}{r_h} \right)^{2n-4}, \tag{3.26}$$

up to numerical factors which are independent of the black hole geometry. Comparing with (3.21), we get agreement only if $n = 7/2$, in other words our extra contribution to the entropy can only arise from a gravitational Lagrangian of the form

$$\mathcal{L}_n = \alpha_{\frac{7}{2}} R^{\frac{7}{2}}. \tag{3.27}$$

For general ξ, the expression for the extra entropy (2.28) is sufficiently complex that we cannot make specific predictions about its source. However, we have seen that it is completely described (modulo the ambiguities in the terms containing logarithms, including, potentially, a term proportional to the logarithm of the event horizon radius) by contributions from the one-loop effective gravitational Lagrangian (2.25) when the black hole is large. This is in accordance with expectations, for when the black hole is large, the semi-classical approximation we are using is valid, and higher-order and non-perturbative corrections to the gravitational Lagrangian should be negligible. Therefore, in this case, the entropy of the quantum scalar field is completely accounted for. Our inclusion of a negative cosmological constant is particularly pertinent in this situation, as large black holes in anti-de Sitter space have a stable Hartle-Hawking state, whereas this state is unstable in asymptotically flat space.

Our calculations indicate that this is not the case for small black holes. However, it should be said that the semi-classical approximation will break down in this limit, so we should not be too surprised that the corrections to the gravitational Lagrangian that are predicted contain unusual terms with the curvature to a fractional power. It will be necessary to appeal to quantum gravity effects in order to get a consistent picture in this case.

In this section, we have considered the simplest possible non-extremal black hole in adS, namely Schwarzschild-adS. However, we anticipate that our conclusions will apply to more general black holes in models with a negative cosmological constant. For example, black holes have been found in $\mathfrak{su}(2)$ Einstein-Yang-Mills theory which possess classically stable gauge field hair in adS [30], precisely when $l^2/r_h^2 \ll 1$, which is the limit in which our semi-classical approach is most valid. In this limit, the geometries of these black holes have the form of small perturbations of the Schwarzschild-adS space-time. Therefore, to leading order in l^2/r_h^2, the finite entropy of a massless quantum scalar field on these black holes would be the same as that we have calculated here for Schwarzschild-adS. Of course, there would be sub-leading corrections due to the additional structure of the black hole geometry, and we hope to return to these matters in a subsequent publication.

2. Extremal black hole

For the Reissner-Nordström-adS black hole, we shall take the approach of "extremalization after quantization" [23], since putting $\beta^{-1} = 0$ in the general expression for the entropy (3.3) will give zero regardless of the black hole geometry. For the RN-adS geometry, the metric function Δ has the form

$$\Delta = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{3.28}$$

where Q is the charge. There are two horizons in this case, an outer (event) horizon at $r = r_h$ and an inner (Cauchy) horizon, at $r = r_h x_-$, where $x_- \leq 1$. When $x_- = 1$, the inner and outer horizons merge and the black hole is extremal. As previously, we use dimensionless variables, and the integral I_r now takes the form

$$I_r = \frac{9}{\Lambda^2 r_h} \int_{1+\hat{\epsilon}}^{\hat{L}} dx\, x^6 \left(x-1\right)^{-2} \left(x - x_-\right)^{-2} \left(x^2 + (1+x_-)x + \zeta\right)^{-2}, \tag{3.29}$$

where the dimensionless parameter ζ is given by

$$\zeta = -\frac{3Q^2}{\Lambda r_h^4 x_-}. \tag{3.30}$$

Although this integral can be performed exactly, it is considerably more complicated than for Schwarzschild-adS. However, matters are simplified because we are interested solely in the extremal limit. Using the dimensionless quantities, and substituting for the inverse temperature β, which is now

$$\beta = -\frac{12\pi}{\Lambda r_h}(1-x_-)^{-1}(2+x_-+\zeta)^{-1}, \tag{3.31}$$

the expression for the entropy (3.3) takes the form:

$$S = -\frac{1}{9720}\Lambda^3 r_h^3 (1-x_-)^3 (2+x_-+\zeta)^3 I_r. \tag{3.32}$$

Therefore we are concerned only with those terms in I_r which remain when multiplied by $(1-x_-)^3$ and the limit $x_- \to 1$ taken. As previously, we shall ignore those terms which vanish as $L \to \infty$ or $\epsilon \to 0$, and also the known divergent terms in ϵ^{-1} or $\log \hat{\epsilon}$ which can be absorbed in an appropriate renormalization of the coupling constants. As in the non-extremal case, there is some ambiguity in the renormalization of the logarithmically divergent term $\log \hat{\epsilon}$, which is dependent upon a choice of a length scale, and could give rise to contributions to the finite entropy which are proportional to the logarithm of the event horizon radius [24]. In this section we shall follow the strategy of section III A 1, and remove by renormalization a term proportional to $\log \hat{\epsilon}$, which will not leave any residual logarithmic terms in the finite entropy. The remaining terms of interest are then:

$$\begin{aligned}\frac{\Lambda^2 r_h}{9}(1-x_-)^3 I_r = {} & \frac{x_-^6}{\mathcal{A}_1} + \frac{\mathcal{A}_2}{\mathcal{A}_3}\log(x_- - 1) \\ & -\frac{\mathcal{A}_4}{\mathcal{A}_5}\frac{(1-x_-)^3}{-4\zeta+1+2x_-+x_-^2}\tanh^{-1}\left(\frac{3+x_-}{\sqrt{-4\zeta+1+2x_-+x_-^2}}\right) \\ & +O(x_- - 1), \end{aligned} \tag{3.33}$$

where we have taken the limits $\epsilon \to 0$, $L \to \infty$ and retained the term involving $\tanh^{-1}$ since it may be of interest in the special case $\zeta = 1$. In (3.33), the factors $\mathcal{A}_i$ are polynomials in x_- and ζ, which are non-zero when either or both of x_- or ζ are equal to unity.

The first term in (3.33) is finite as $x_- \to 1$, and, for this value of x_-, we have

$$\mathcal{A}_1 = (\zeta+3)^2. \tag{3.34}$$

The second term has a logarithmic singularity as $x_- \to 1$. However, as has already been observed with other logarithmic terms, the coefficient of $\log(x_- - 1)$ in (3.33) is exactly the same as the coefficient of the $\log \hat{\epsilon}$ term in the integral. Therefore we may absorb the $\log(x_- - 1)$ term in an appropriate renormalization of the coupling constants. The third term vanishes in the limit $x_- \to 1$ unless $\zeta = 1$ also. The case $x_- = \zeta = 1$ can be dealt with in two ways: firstly, we can set $\zeta = 1$ for all x_- and then let $x_- \to 1$ subsequently; or secondly we can set $\zeta = x_-$ and then let $x_- \to 1$. Proceeding either way, the third term in (3.33) vanishes in the limit $x_- \to 1$.

We may now take the limit $x_- \to 1$ in (3.33) and substitute in the expression for the entropy (3.32). The answer is:

$$S_{\text{extra}} = -\frac{1}{1080}\Lambda r_h^2(\zeta+3) = \frac{1}{360}\left[\frac{Q^2}{r_h^2} - \Lambda r_h^2\right], \tag{3.35}$$

where we have re-introduced the original geometric parameters. The final expression for the entropy for an extremal RN-adS black hole is rather more simple than the corresponding expression (3.13) for Schwarzschild-adS. Furthermore, the expression (3.35) is exactly proportional to the entropy coming from the $R_{\rho\sigma}R^{\rho\sigma}$ term in the effective Lagrangian (2.25). Therefore the entropy in this case comes purely from higher-order corrections to the gravitational Lagrangian, and may be absorbed in a suitable redefinition of the coupling constants:

$$b_B \to b_B - \frac{1}{2880\pi}. \tag{3.36}$$

It is perhaps surprising that our expression for the entropy (3.35) is so much simpler than that for Schwarzschild-adS, and that the simple renormalization (3.36) works for all values of r_h in this case. However, we are dealing in this situation with an extremal black hole, which has zero temperature, and therefore there are no thermal excitations of the quantum field modes. In the case of Schwarzschild-adS, we considered the limit of large black holes, since it is in this limit that the back-reaction on the geometry due to Hawking radiation is expected to be small, and, further, the Hartle-Hawking state representing the thermal equilibrium between the black hole and the thermal heat bath surrounding it is stable. Since the temperature is zero for an extremal black hole, this is not an issue, and so our approach is valid for all values of l/r_h.

B. Massive scalar field

We close this section by considering a massive rather than a massless scalar field, for an extremal black hole. We anticipate that the inclusion of a scalar field mass in the calculation of entropy for a non-extremal black hole will not significantly alter the qualitative results of section III A 1, but it does greatly increase the computational complexity. The integral corresponding to I_r (3.2) for a massive quantum scalar field on an S-adS black hole is not readily tractable in general. Accordingly, it also seems reasonable to suppose that the inclusion of scalar field mass will also not change the qualitative results for the entropy on an extremal black hole geometry calculated via an "extremalization after quantization" approach. It may, however, make a difference to the "extremalization before quantization" calculation, which we consider in this section.

For a quantum scalar field of mass μ, the free energy is given by (2.13), which gives, for an extremal RN-adS black hole

$$F = -\frac{6}{\pi\Lambda^2 r_h}\int \frac{dE}{e^{\beta E}-1}\int_{1+\hat{\epsilon}}^{\hat{L}} dx\, x^6 (x-1)^{-4}(x^2+2x+\zeta)^{-2} \times \left[E^2 + \frac{\mu^2\Lambda r_h^2}{3} x^{-2}(x-1)^2(x^2+2x+\zeta)\right]^{\frac{3}{2}}, \tag{3.37}$$

where we are using dimensionless variables as in the previous subsection. The r integral is not readily tractable for general ζ, so we shall consider only the simplest case, when $\zeta = 1$. Setting

$$\hat{E} = E\sqrt{\frac{-3}{\mu^2\Lambda r_h^2}}, \qquad \hat{\beta} = \beta\frac{\sqrt{-\Lambda}}{\sqrt{3}}\mu r_h, \tag{3.38}$$

the integral (3.37) has the form

$$F = -\frac{2}{3\pi}\mu^4 r_h^3 \int \frac{d\hat{E}}{e^{\hat{\beta}\hat{E}}-1}\int_{1+\hat{\epsilon}}^{\hat{L}} dx\, x^6(x-1)^{-4}(x+1)^{-4}\left[\hat{E}^2 - \frac{1}{x^2}(x-1)^2(x+1)^2\right]^{\frac{3}{2}}. \tag{3.39}$$

The r integral can be performed exactly. Inserting the limit $\hat{L}\to\infty$ gives no contribution, since the argument of the square root in (3.39) must be positive, so that when $x \gg 1$, it must be the case that $\hat{E} \gg 1$, which will give a vanishing $\hat{E}$ integral. We already know from section II D that any terms which diverge as $\hat{\epsilon}\to 0$ will not contribute to the entropy. Therefore, inserting the lower limit $x = 1+\hat{\epsilon}$ gives, ignoring any divergent terms,

$$F = \frac{1}{144\pi}\mu^4 r_h^3 \int \frac{d\hat{E}}{e^{\hat{\beta}\hat{E}}-1}\left\{43\hat{E}^3 - 20\hat{E} + \left(48 + 72\hat{E}^2\right)\tan^{-1}\left(\frac{\hat{E}}{2}\right) + \left(15\hat{E}^3 + 108\hat{E}\right)\left[\log 2 - \frac{1}{2}\log\left(1+\frac{2}{\hat{E}^2}\right)\right]\right\} + O(\hat{\epsilon}). \tag{3.40}$$

With the standard integrals (2.18) it can be seen that the terms containing just $\hat{E}^3$ or $\hat{E}$ give, after performing the $\hat{E}$ integral, terms proportional to β^{-4} or β^{-2} respectively. When we then calculate the entropy, via (2.20), we end up with terms proportional to β^{-3} or β^{-1}. Since we have an extremal black hole, these terms will all vanish when we substitute the "on shell" temperature $\beta^{-1} = 0$. Then (3.40) reduces to

$$F = \frac{1}{144\pi}\mu^4 r_h^3 \int \frac{d\hat{E}}{e^{\hat{\beta}\hat{E}}-1}\left\{\left(48+72\hat{E}^2\right)\tan^{-1}\left(\frac{\hat{E}}{2}\right) - \frac{1}{2}\left(15\hat{E}^3 + 108\hat{E}\right)\log\left(1+\frac{2}{\hat{E}^2}\right)\right\}. \tag{3.41}$$

We shall now consider the integrals in the above equation in turn.

Firstly,

$$\int \frac{d\hat{E}}{e^{\hat{\beta}\hat{E}}-1}(48+72\hat{E}^2)\tan^{-1}\left(\frac{\hat{E}}{2}\right) = \frac{2\pi}{\hat{\beta}}\left[48\int\frac{du}{e^{2\pi u}-1}\tan^{-1}\left(\frac{u}{q}\right) + \frac{288\pi^2}{\hat{\beta}^2}\int\frac{du}{e^{2\pi u}-1}u^2\tan^{-1}\left(\frac{u}{q}\right)\right], \tag{3.42}$$

where

$$u = \frac{\hat{\beta}\hat{E}}{2\pi}, \qquad q = \frac{\hat{\beta}}{\pi} \tag{3.43}$$

These are now standard integrals, which can be written for general q in terms of gamma functions. However, we are interested only in values of $q \gg 1$. Therefore we may simplify by using the asymptotic form of the gamma function for large q. This gives

$$\begin{aligned} \int \frac{du}{e^{2\pi u} - 1} \tan^{-1}\left(\frac{u}{q}\right) &= \frac{1}{12q} + O\left(\frac{1}{q^3}\right); \\ \int \frac{du}{e^{2\pi u} - 1} u^2 \tan^{-1}\left(\frac{u}{q}\right) &= \frac{1}{60q} + O\left(\frac{1}{q^2}\right). \end{aligned} \tag{3.44}$$

Therefore (3.42) is, to leading order, proportional to β^{-2}, and this will give vanishing contribution to the entropy. The only remaining term which could contribute to the free energy is now

$$\begin{aligned} &\int \frac{d\hat{E}}{e^{\hat{\beta}\hat{E}} - 1}\left(15\hat{E}^3 + 108\hat{E}\right)\log\left(1 + \frac{2}{\hat{E}^2}\right) \\ &\qquad = \int \frac{dv}{e^v - 1}\left(\frac{15}{\hat{\beta}^4}v^3 + \frac{108}{\hat{\beta}^2}v\right)\left[\log\left(1 + \frac{v^2}{2\hat{\beta}^2}\right) + \log 2\hat{\beta}^2 - \log v^2\right], \end{aligned} \tag{3.45}$$

where $v = \hat{\beta}\hat{E}$. In this form, it is clear that this remaining integral has leading behaviour $\hat{\beta}^{-2}\log\hat{\beta}$ for $\hat{\beta} \gg 1$, so again there is no contribution to the entropy as $\beta^{-1} \to 0$.

We therefore conclude that the exact entropy, for a massive quantum scalar field, is identically zero in this approach.

IV. CONCLUSIONS

In this article we have calculated the entropy of a quantum scalar field in a thermally excited state outside the event horizon of a black hole using the "brick wall" model of 't Hooft [2]. We considered spherically symmetric black holes in asymptotically anti-de Sitter (adS) space, which means that the theory is infra-red convergent. Ultra-violet divergences remain due to the infinite number of modes close to the event horizon, and we regulate these by using a cut-off a proper distance $\tilde{\epsilon}$ away from the event horizon (we focus on non-extremal black holes). We showed that the divergences can all be absorbed into a suitable renormalization of the coupling constants in the one-loop effective gravitational Lagrangian, yielding a finite entropy. This is in agreement with similar calculations in the literature performed via different methods.

However, the renormalization of the logarithmic divergences is dependent upon the choice of a length scale, and can lead to a contribution to the entropy which is proportional to the logarithm of the event horizon radius. Terms of this form have been found in other approaches to black hole entropy. However, the picture here is not entirely clear, and one natural choice of the length scale does not give any term of this form. We have argued that for black holes in adS, this choice of length scale, namely that set by the cosmological constant, is in fact the most reasonable to consider for this theory. However, we consider this to be an open question requiring further investigation, particularly for general black holes not in adS, where it is not clear what might be an appropriate choice of length scale.

We next considered the finite entropy (modulo these logarithmic ambiguities), for the particular case of the Schwarzschild-adS black hole and a massless quantum scalar field. For large black holes, the entropy so produced is precisely accounted for from the one-loop Lagrangian. We conjecture that this will also be the case for more general black holes, but leave this question for future investigations. However, this is not the case for small black holes and there are indications from dimensional arguments that non-perturbative quantum gravity corrections may be important in this limit.

For extremal black holes, there are two approaches that can be taken. The first is "extremalization after quantization", which gives the same qualitative results as in the non-extremal case. In other words, both the divergent and finite contributions (apart from logarithmic terms) to the entropy are non-zero and arise from the effective Lagrangian. The second approach, "extremalization before quantization", yields an "on-shell" entropy which has no divergences. The finite entropy vanishes for a massless scalar field, and also for a particular black hole with a massive scalar field. We anticipate that this result will also apply to more general extremal black holes.

Although our understanding of black hole entropy has greatly increased over the nearly thirty years the concept has been around, many open questions remain and its microscopic origin is not completely known. The semi-classical

approach advocated in this paper is straightforward both computationally and conceptually, and readily produces finite answers. Our results in this paper reveal an encouraging consistency in this approach, apart from an ambiguity in logarithmic terms, although we have argued that there is a natural choice of length scale which does not produce any contribution to the entropy proportional to the logarithm of the event horizon radius. Namely, the entropy of the quantum field on the classical background (apart from these logarithmic terms) is entirely accounted for by the additional terms in the gravitational Lagrangian that must be included in order to renormalize the semi-classical theory. In the large black hole regime where this theory applies, no new terms arise due to this scenario. We anticipate that this result can be extended to more general black hole geometries, and quantum fields of non-vanishing spin. In the present paper, we have considered only black holes in adS space, motivated partly by the calculational simplicity afforded by the regularization of the contribution to the entropy from infinity in this case. We consider that the main qualitative conclusions of this article will also apply to black holes not in adS space. However, the details will need to be studied separately, particularly with regard to the choice of cut-off. We hope to return to these questions in the near future.

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[1] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973).
[2] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).
[3] R. B. Mann, L. Tarasov, and A. Zelnikov, Class. Quant. Grav. **9**, 1487 (1992).
[4] L. Susskind and J. Uglum, Phys. Rev. **D50**, 2700 (1994).
[5] J. L. F. Barbón and R. Emparan, Phys. Rev. **D52**, 4527 (1995).
[6] S. P. de Alwis and N. Ohta, Phys. Rev. **D52**, 3529 (1995).
[7] S. N. Solodukhin, Phys. Rev. **D51**, 618 (1995).
[8] N. E. Mavromatos and E. Winstanley, Phys. Rev. **D53**, 3190 (1996).
[9] J.-G. Demers, R. Lafrance, and R. C. Myers, Phys. Rev. **D52**, 2245 (1995).
[10] D. V. Fursaev and S. N. Solodukhin, Phys. Lett. **B365**, 51 (1996).
[11] T. Shimomura, Phys. Lett. **B480**, 207 (2000).
[12] S. N. Solodukhin, Phys. Rev. **D54**, 3900 (1996).
[13] S. P. Kim, S. K. Kim, K.-S. Soh, and J. H. Yee, Int. J. Mod. Phys. **A12**, 5223 (1997).
[14] S. Mukohyama and W. Israel, Phys. Rev. **D58**, 104005 (1998).
[15] V. P. Frolov, D. V. Fursaev, and A. I. Zelnikov, Phys. Lett. **B382**, 220 (1996).
[16] J. Ho, W. T. Kim, and Y.-J. Park, Class. Quant. Grav. **14**, 2617 (1997).
[17] S. P. Kim, S. K. Kim, K.-S. Soh, and J. H. Yee, Phys. Rev. **D55**, 2159 (1997).
[18] S. N. Solodukhin, Phys. Rev. **D56**, 4968 (1997).
[19] S. W. Hawking and D. N. Page, Comm. Math. Phys. **87**, 577 (1983).
[20] A. Ghosh and P. Mitra, Phys. Rev. Lett. **73**, 2521 (1994).
[21] R.-G. Cai and Y.-Z. Zhang, Mod. Phys. Lett. **A11**, 2027 (1996).
[22] A. Ghosh and P. Mitra, Phys. Lett. **B357**, 295 (1995).
[23] A. Ghosh and P. Mitra, Phys. Rev. Lett. **78**, 1858 (1997).
[24] R. B. Mann and S. N. Solodukhin, Nucl. Phys. **B523**, 293 (1998).
[25] C. Kiefer and J. Louko, Ann. Phys. (Leipzig) **8**, 67 (1999).
[26] R. K. Kaul and P. Majumdar, Phys. Rev. Lett. **84**, 5255 (2000).
[27] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series and products* (Academic Press, London, 1965).
[28] N. D. Birrell and P. C. W. Davies, *Quantum fields in curved space* (Cambridge University Press, Cambridge, 1982).
[29] R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics* (University of Chicago Press, Chicago, 1994).
[30] E. Winstanley, Class. Quant. Grav. **16**, 1963 (1999).